Scientific Report [*]

Bilevel programming

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1 Purpose of the STSM

In the beginning of 2012, we started the study of a fundamental integer bilevel optimization problem proposed in [3]. The mathematical formulation is a natural extension of the classic 0–1 knapsack problem to two levels. In short, the problem describes a Stackelberg game where the upper level interdicts a subset of the lower level's knapsack items.

The goal of this STSM was to intensively discuss the algorithmic ideas developed so far to solve the referred bilevel knapsack optimization problem. Moreover, from this meetings, it was expected to result a clear and formal description of our theoretical and practical contributions in this context.

2 Description of the work carried out during the STSM

Since the beginning of our collaboration, we tried to always have an updated draft paper of all our achievements in the field of bilevel optimization. Therefore, in the first meetings of the STSM, we improved that draft by adding and correcting theoretical results.

Then, we decided which computations would be interesting to perform in order to see the effectiveness of our approach to solve the bilevel knapsack problem. In this way, it was done a detailed analysis of our algorithm by solving random instances.

Finally, we compared the performance of our algorithm with a natural cutting plane approach and the method of [3].

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3 Description of the main results obtained

The bilevel knapsack problem in which we have been concentrated is the following

$$
\begin{aligned}
(DN) \qquad & \underset{(x,y)\in\{0,1\}^n\times\{0,1\}^n}{\text{minimize}} \sum_{i=1}^{n} p_i y_i && \text{(1a)}\\
& \text{subject to} \qquad \sum_{i=1}^{n} v_i x_i \le C_u && \text{(1b)}\\
& \qquad \text{where } y_1,\dots,y_n \text{ solves the follower's problem} && \\
& \qquad \underset{y\in\{0,1\}^n}{\text{maximize}} \sum_{i=1}^{n} p_i y_i && \text{(1c)}\\
& \qquad \text{s.t.} \quad \sum_{i=1}^{n} w_i y_i \le C_l && \text{(1d)}\\
& \qquad y_i \le 1 - x_i \quad \text{for } 1 \le i \le n. && \text{(1e)}
\end{aligned}
$$

Our final algorithm proposal, the Caprara-Carvalho-Lodi-Woeginger Algorithm (CCLW), is the merging of several theoretical results that allow a significant reduction of computational times to find the an optimal solution.

The bottom line of CCLW is the upper bound to DN that results by relaxing the integrality of the lower level variables y in DN. We prove that this upper bound can be computed by solving a mixed integer programming problem

$$
\begin{aligned}
(MIP^1) \qquad & \underset{x\in\{0,1\}^n, z\in[0,\infty)^{n+1}, u\in[0,\infty)^n}{\text{minimize}} \quad z_0 C_l + \sum_{i=1}^{n} u_i && \text{(2a)}\\
& \text{subject to} \qquad \sum_{i=1}^{n} v_i x_i \le C_u && \text{(2b)}\\
& \qquad u_i \ge 0 \qquad \text{for } 1 \le i \le n && \text{(2c)}\\
& \qquad u_i \ge z_i - p_i x_i \qquad \text{for } 1 \le i \le n && \text{(2d)}\\
& \qquad w_i z_0 + z_i \ge p_i \qquad \text{for } 1 \le i \le n. && \text{(2e)}
\end{aligned}
$$

In this way, a feasible solution x^1 to the upper level is computed. In practice, this solution happens to be in general the optimal one. This step is followed by solving the 0–1 knapsack problem (lower level problem) for the upper level solution found x^1. See Figure 1.

All the algorithms for (mixed) integer bilevel optimization problems in the literature are enumerative approaches in the sense that several upper level solutions are computed until the end of the algorithm. CCLW is not an exception, although its way of doing this enumeration is completely new.

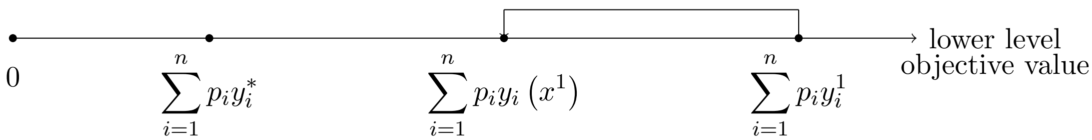

Figure 1: Illustration of the upper bounds to DN, where (x^*, y^*) is an optimal solution to DN, $\left(x^1, y^1\right)$ is the optimal solution to MIP^1 and $\left(x^1, y\left(x^1\right)\right)$ is the corresponding lower level optimal solution for x^1.

CCLW is an iterative method. In each iteration k of CCLW, it is computed a new upper level feasible solution x^k by solving MIP^1 with additional constraints. This additional constraints enable us to reduce significantly the number of iterations and constitute one of the main results of our work.

The constraints to be added to MIP^1 in each iteration allow us to avoid the computation of upper level feasible solutions that will not improve (reduce) the current upper bound value, OPT, to DN and therefore, are uninteresting. In this context we proved the validity of the *cutting plane constraints*

$$\sum_{i=1}^{n} y_i \left(x^k\right) p_i \left(1 - x_i\right) \leq OPT - 1.$$

This constraint can be interpreted as follows: if OPT is not the optimal value of DN, the upper level must be able to reduce it since that is its goal.

The final essential ingredient of CCLW is the *strong constraint*

$$z_0 C_l + \sum_{i=1}^{n} u_i - z_0 w_{max} \leq OPT - 1$$

where $w_{max} = \underset{i=1,\ldots,n}{\text{maximize}}\, w_i$. We proved that this constraint is valid. The *strong constraint* traduces some how the lower level optimal reaction to an upper level solution allowing us to save the enumeration of uninteresting solutions.

Through our computational results, it was clear the importance of the two constraints described above to have an approach superior to the ones available in the literature. Moreover, a generalization of this constraints to an approach able to solve general bilevel interdiction problems seems easy.

4 Future collaboration with the host institution

Our collaboration has been very fruitful, thus we are searching for a real-world application of integer bilevel optimization to apply our knowledge in this field. Namely, bilevel optimization formulations have been extensively used in planning and protection of infrastructures (see [4] and [1]) which could be interesting in the context of efficient and robust energy networks. Another interesting application of a bilevel formulation appears in the context of short-term electricity markets, see [2].

In the meanwhile, we are studying the generalization of our results to general interdiction bilevel optimization problems.

5 Foreseen publications/articles resulting from the STSM

From this STSM, it is going to result a paper entitled: *Bilevel knapsack with interdiction constraints*. Currently, the authors of this paper are making a careful final reading of it.

References

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