Nonlinear Mixed-Integer Optimization

MINO/COST SPRING SCHOOL ON OPTIMIZATION

Jon Lee — Andrea Lodi

University of Michigan — University of Bologna

10 April 2014

Outline for the Day

- 9:15-10:15. lecture (Lee): Introduction to MINLP // Complexity of MINLP: Hardness and polynomial tractability
- 10:15-11:00. coffee break
- 11:00-12:00. lecture (Lodi): General-purpose algorithms for convex and non-convex MINLP
- 12:00-14:15. lunch
- 14:15-15:00. lecture (Lee): Non-convex quadratic MINLP
- 15:00-15:30. coffee break
- 15:30-16:15. lecture (Lodi): Software and computational advances
- 16:15-16:30. short break
- 16:30-17:00. problem session

Mixed Integer Nonlinear Programming algorithms

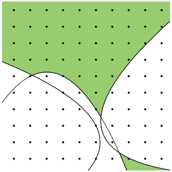
Andrea Lodi (and Pierre Bonami)

DEI, University of Bologna

MINO-COST Training School – Klagenfurt – April 10, 2014

"The mother of all deterministic optimization problems" [Lee, 2008]

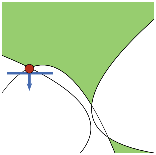
$$
\begin{array}{lll}
\min & f(x) & \\
\text{s.t.} & g_i(x) \leq 0 & i = 1, \dots, m \\
& x \in X & \\
& x_j \in \mathbb{Z} & j = 1, \dots, p \\
& l_j \leq x_j \leq u_j & j = 1, \dots, p
\end{array}
\qquad \text{(MINLP)}
$$

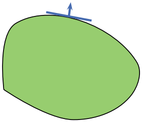


- $X \subseteq \mathbb{R}^n$ polyhedral.
- f and $g_i : X \rightarrow \mathbb{R}$, $i = 1, \dots, m$, continuous, differentiable.

Nonlinear Programming (NLP)

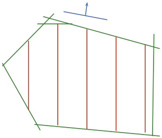
$p = 0$: local optima. $\quad + f$ and g_i convex $\Rightarrow$ global optima.





Mixed-Integer linear programming (MILP)

- f linear, $m = 0$, $p > 0$



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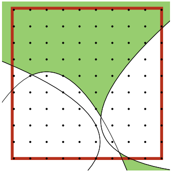


- **Solvable, in general,** l_j, u_j **finite**.

Mixed Integer Convex Programming

Assume that the continuous relaxation is a convex optimization problem.

- f is a convex function.
- g_i are either convex function or describe a convex feasible region (for example, second order cone constraints: $\sum x_j^2 \leq x_0^2$)

Mixed Integer Nonlinear Programming

Do not assume any convexity on f or g_i.

- Continuous relaxation is NP-hard to solve in general.
- Remark: if l_j and u_j are finite integer variable can be seen as a continuous satisfying

$$(x_j - l_j)(x_j - l_j - 1)\dots(x_j - u_j) = 0$$

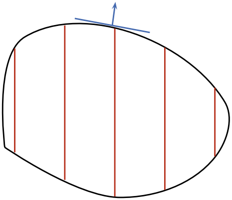
Application	nonlinear	discrete
Portfolio optimization	Risk, utility, robustness	number of assets, min investment
[Bienstock, 1996, Bonami and Lejeune, 2009, Vielma et al., 2008]		
Chemical plant design	Chemical reactions	what to install
[Duran and Grossmann, 1986, Flores-Tlacuahuac and Biegler, 2007]		
Block Layout Design	Spatial constraints	what to layout
[Castillo et al., 2005]		
Networks with delays	Delay as function of traffic	Path, flows
[Boorstyn and Frank, 1977, Ameur and Ouorou, 2006]		
Location with stochastic services	Demands	location model
[Elhedhli, 2006]		
TSP with neighborhoods (Robotics)	Definition of ngbh.	TSP
[Gentilini et al., 2013]		

Application	nonlinear	discrete
Petrochemical	Blending, pooling	Which process
[Haverly, 1978]		
Gas/Water networks	Pressure loss	Network topology
[Bragalli et al., 2011]		
Nuclear Reactor reloading	reactions	What to reload
[Quist et al., 1999]		
Airplane trajectory optimization	aerodynamics	waypoints, colision avoidance,...
[Cafieri and Durand, 2013, Soler et al., 2013]		
Mixed Integer Optimal control	DE	discrete controls
[Sager, 2005, 2012]		
Countless more	...	...
see for example [Belotti et al., 2013]		

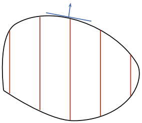
- The convex case
  - Main algorithmic approaches.
  - Glimpse at computation.
- A step into nonconvexity
  - MIQP with CPLEX 12.6.
  - Basic setup of a spatial branch-and-bound.
  - Computational results.
- Arbitrary selection of more advanced topics
  - Separability.
  - Disjunctive Cuts.
- Conclusions
- Problem Session
- Bibliography

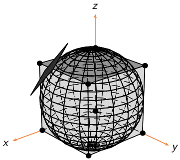
The mixed integer convex program

$$
\begin{array}{lll}
\textbf{min} & c^T x & \\
\text{s.t.} & g_i(x) \leq 0 & i = 1, \dots, m \\
& x \in X & \\
& x_j \in \mathbb{Z} & j = 1, \dots, p \\
& l_j \leq x_j \leq u_j & , j = 1, \dots, p
\end{array}
\quad \text{(MICP)}
$$



- $g_i : X \to \mathbb{R}$, $i = 1, \dots, m$, convex, differentiable.
- Assume linear objective. If necessary, add var $\alpha \in \mathbb{R}$ and $\min \alpha$ with $f(x) \leq \alpha$ as a constraint.





Fundamental property is convexity of the continuous relaxation, which can be efficiently solved.

1. NLP Branch-and-bound [Gupta and Ravindran, 1985].
2. Outer Approximation Algorithm [Duran and Grossmann, 1986]. Builds an MILP equivalent of (MICP)
3. LP/NLP branch-and-cut [Quesada and Grossmann, 1992].

Straightforward generalization of main MILP algorithm:

- solve an NLP at each node of the tree.

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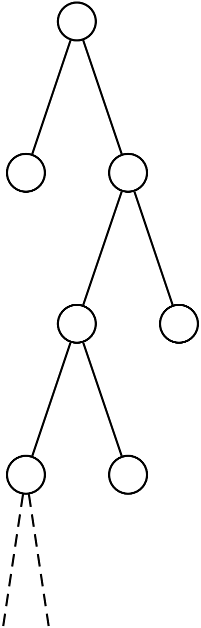
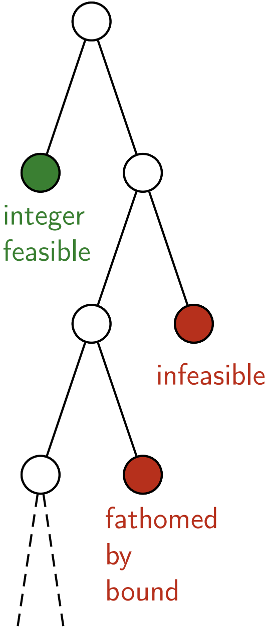
- Branch on variables with fractional value.



Straightforward generalization of main MILP algorithm:

- solve an NLP at each node of the tree.
- Branch on variables with fractional value.
- Prune by infeasibility, bounds and integer feasibility.



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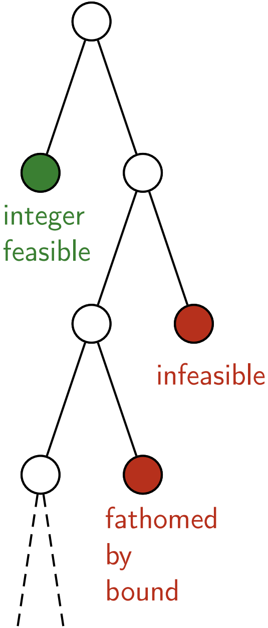
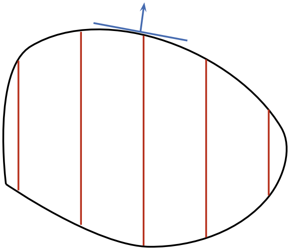
Main issues

- Warm-starting of NLP solves.
- Difficulty of reusing MILP technologies.



Outer Approximation [Duran and Grossmann, 1986]



$$\min \quad c^T x$$

s.t.

$$g_i(x) \le 0 \qquad i = 1, \dots, m,$$

$$x_j \in \mathbb{Z} \qquad j = 1, \dots, p.$$

Idea: Take first-order approximations of constraints at different points and build an equivalent MILP.

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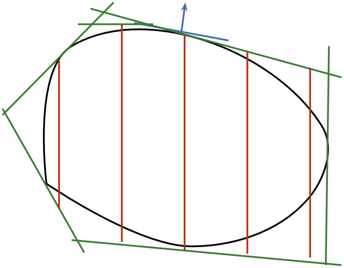

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Idea: Take first-order approximations of constraints at different points and build an equivalent MILP.

$$\min \quad c^T x$$

s.t.

$$g_i(\overline{x}^k) + \nabla g_i(\overline{x}^k)^T (x - \overline{x}^k) \leq 0 \qquad i = 1, \ldots, m,\ k = 1, \ldots, K$$

$$x_j \in \mathbb{Z} \qquad j = 1, \ldots, p.$$

Given $\hat{x} \in \mathbb{Z}^p$:

fixed NLP (NLP($\hat{x}$))

$$\begin{aligned} & \min\ c^T x \\ & s.t. \\ & g_i(x) \le 0 \qquad i = 1, \dots, m \\ & x \in X \qquad\qquad (\text{NLP}(\hat{x})) \\ & x_j = \hat{x}_j \qquad j = 1, \dots, p. \end{aligned}$$

If $\hat{x} \in \mathbb{Z}^p$, and feasible: gives an upper bound.

fixed feasibility subproblem

$$\begin{aligned} & \min \sum_{i=1}^{m} w_i \max\{0, g_i(x)\} \\ & \text{s.t.} \\ & x \in X, \qquad\qquad (\text{NLPF}(\hat{x})) \\ & x_j = \hat{x}_j,\ j = 1, \dots, p \end{aligned}$$

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Remark: If (NLP($\hat{x}$)) is infeasible, NLP software will typically return a solution to (NLPF($\hat{x}$)). By abuse, always say solution to (NLP($\hat{x}$))

For each $\hat{x}^k \in K = \text{Proj}_{1,\dots,p}(X) \cap \mathbb{Z}^p$, let $\overline{x}^k$ be an optimal solution to (NLP($\hat{x}$)).

Theorem ([Duran and Grossmann, 1986])

If $X \neq \emptyset$, f and g are convex, continuously differentiable, and a constraint qualification holds for each $\overline{x}^k$ then

$$\begin{aligned} \min \quad & c^T x \\ & g_i(\overline{x}^k) + \nabla g_i(\overline{x}^k)^T (x - \overline{x}^k) \leq 0 \quad i = 1, \dots, m, \hat{x}^k \in K, \\ & x \in X,\ x_j \in \mathbb{Z},\ j = 1, \dots, p. \end{aligned}$$

has the same optimal value as (MICP).

Generate MILP equivalent by constraint generation.

- Initialize with one set of linearizations.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \\ & g_i(\overline{x}^0) + \nabla g_i(\overline{x}^0)^T (x - \overline{x}^0) \leq 0, \quad i = 1, \ldots, m, \quad , \\ & x \in X,\ x_j \in \mathbb{Z},\ j = 1, \ldots, p. \end{array} \qquad (\text{OA}(\mathcal{K}))$$

Where x^0 is the solution to the continuous relaxation:
$\min\{c^T x : x \in X,\ g_i(x) \leq 0, i = 1, \ldots, m\}$

Generate MILP equivalent by constraint generation.

- Initialize with one set of linearizations.
- Enrich iteratively the set of linearizations $\mathcal{K}$.

$$
\begin{aligned}
&\min \quad c^T x \\
&\text{s.t.} \\
&g_i(\overline{x}^k) + \nabla g_i(\overline{x}^k)^T (x - \overline{x}^k) \le 0, \qquad \begin{array}{c} i = 1, \dots, m, \\ \hat{x}^k \in \mathcal{K} \end{array}, \qquad (\text{OA}(\mathcal{K})) \\
&x \in X,\ x_j \in \mathbb{Z},\ j = 1, \dots, p.
\end{aligned}
$$

Where $\hat{x}^k$ is a solution to $(\text{OA}(\mathcal{K}))$ and, for $k = 1, \dots, |\mathcal{K}|$, $\overline{x}^k$ is the solution to $(\text{NLP}(\hat{x}))$.

Generate MILP equivalent by constraint generation.

- Initialize with one set of linearizations.
- Enrich iteratively the set of linearizations $\mathcal{K}$.

Convergence

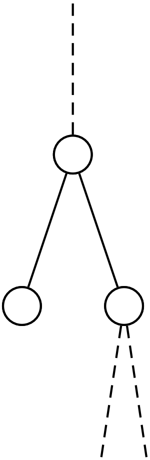
At each iteration:

- (OA($\mathcal{K}$)) gives a lower bound,
- If feasible, (NLP($\hat{x}$)) gives an upper bound.

The theorem guarantees that the two bounds converge in finite # of iterations.

OA can be embedded in a single tree search.

- Start solving the same initial MILP by branch and bound.
- At each integer feasible node:



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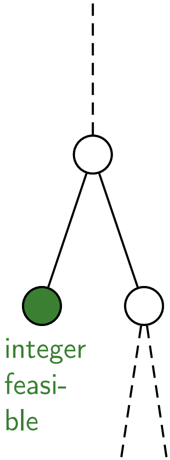
- Start solving the same initial MILP by branch and bound.
- At each integer feasible node:
  1. solve (NLP($\hat{x}$)), and enrich the set of linearizations.
  2. Resolve the LP relaxation of the node with the new cuts.
  3. Repeat as long as node is integer feasible.



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  1. solve (NLP($\hat{x}$)), and enrich the set of linearizations.
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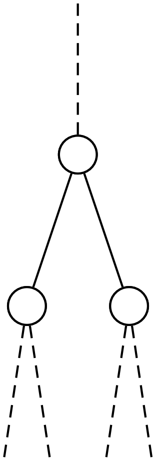
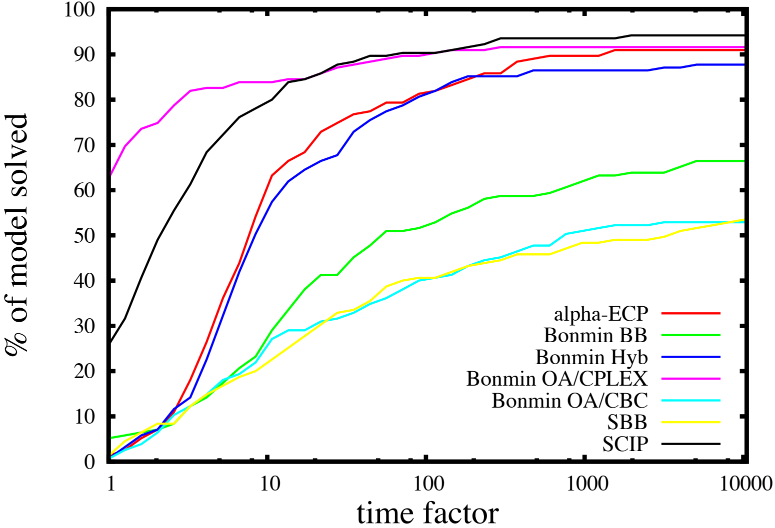
  3. Repeat as long as node is integer feasible.
- Never prune by integer feasibility.



Solvers for Mixed Integer Convex Programs

Solver	Reference	Algorithm(s)
Dicopt		OA
MINLP_BB	[Leyffer, 1998]	NLP BB
SBB	[Bussieck and Drud, 2001]	NLP BB
α-ECP	[Westerlund and Lundqvist, 2005]	ECP (variant of OA)
Bonmin	[Bonami et al., 2008]	NLP BB, OA, LP/NLP
FilMINT	[Abhishek et al., 2010]	LP/NLP
KNITRO	[Byrd et al., 2006]	NLP BB, LP/NLP
SCIP	[Vigerske, 2013]	LP/NLP

Comparison of solvers in GAMS [Vigerske, 2012]



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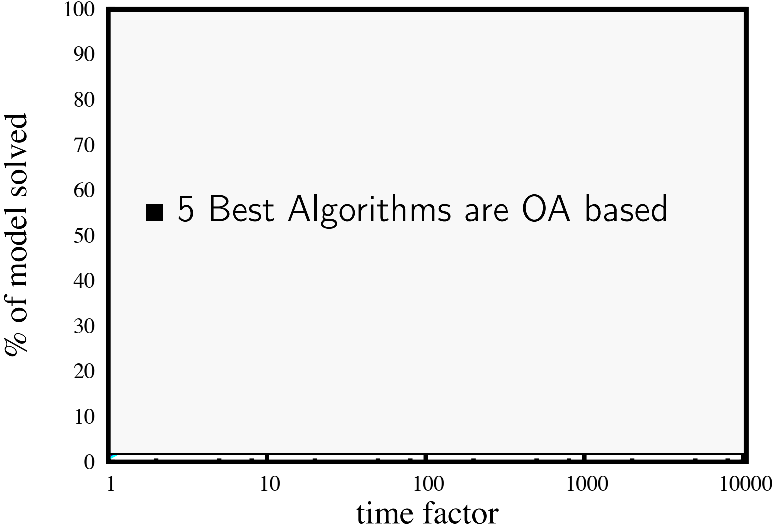

- 5 Best Algorithms are OA based

- Bonmin's OA using CPLEX seems the best algorithm overall.
  - It is also the simplest: a loop calling CPLEX (MILP) and Ipopt (NLP) alternatively as black boxes.
  - Improves with CPLEX.
- Bonmin's Hyb is in the pack of relatively good solvers
  - own variant of LP/NLP BB.
  - Reuse CBC infrastructure, LP solver, Cuts, MIP presolve.
  - Improves at a slower pace.
- Bonmin's BB clearly behind.
  - pure NLP based branch-and-bound. Does not reuse much from Cbc. Everything specifically tailored.
  - Better implementation exists that should be on par with Hyb.

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(MI)QP

$$
\begin{aligned}
\min \quad & \frac{1}{2} x^T Q x + c^T x \\
s.t. \quad & \\
& Ax = b && \qquad \text{(MIQP)} \\
& x_j \in \mathbb{Z} \qquad j = 1, \ldots, p \\
& l \leq x \leq u
\end{aligned}
$$

(with Q symmetric),

Andrea Lodi, University of Bologna

$$\min \frac{1}{2}x^T Qx + c^T x$$

$$s.t.$$

$$Ax = b \qquad \text{(MIQP)}$$

$$x_j \in \mathbb{Z} \qquad j = 1, \ldots, p$$

$$l \leq x \leq u$$

(with Q symmetric),

History of MIQP with CPLEX

class	p	Q	algorithme	V. (Year)
Convex QP	0	$\succeq 0$	barrier	4.0 (1995)
–	–	–	QP simplex	8.0 (2002)
convex MIQP	> 0	$\succeq 0$	B&B	8.0 (2002)
nonconvex QP	0	$\not\succeq 0$	barrier (local)	12.3 (2011)
–	–	–	spatial B&B (global)	12.6 (2013)
nonconvex MIQP	> 0	$\not\succeq 0$	spatial B&B (global)	12.6 (2013)

Let $G = (N, E)$ be a graph and Q be the incidence matrix of G. The optimal value of:

$$\min \frac{1}{2} x^T Q x$$

$$s.t.$$

$$\sum_{j \in N} x_j = 1$$

$$x \geq 0.$$

is $\frac{1}{2}\left(1 - \frac{1}{\chi(G)}\right)$ where $\chi(G)$ is the clique number of G [Motzkin and Straus, 1965],

- $\Rightarrow$ QP is NP-hard
- More generally QPs on the simplex (general Q) can be solved by a nonlinear maximum clique algorithm [Scozzari and Tardella, 2008].

- Primal Dual Interior Point Algorithm.
- Available since IBM CPLEX 12.3.
- Not enabled by default, if Q is indefinite CPLEX will return `CPXERR_Q_NOT_POS_DEF`.
- Activated by setting the option `solution target` to 2 (or `CPX_SOLUTIONTARGET_FIRSTORDER`).
- Approach used by Ipopt but no need for
  - Feasibility restoration
  - Second order correction
  - Filter
- Own implementation of indefinite factorization.

Global (MI)QP

- Activated by setting `solution target` to 3 (or `CPX_SOLUTIONTARGET_OPTIMALGLOBAL`).
- Note: previous versions could already solve some nonconvex MIQPs (pure 0-1 QPs, convex after presolve...)

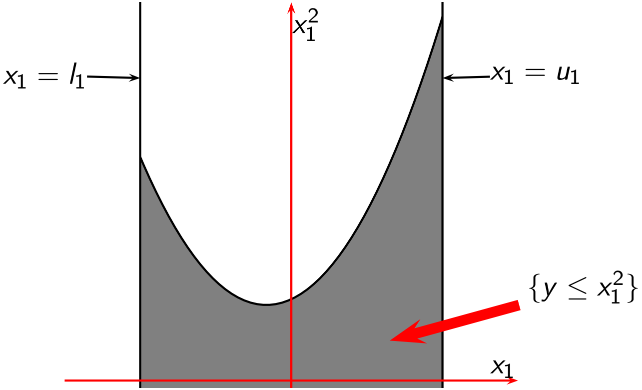
Notes on complexity

- Checking if a feasible solution is not a local minimum is coNP-Complete.
- Checking if a nonconvex QP is unbounded is NP-complete.

B&B spatial

- Establish a convex (easily solvable) relaxation.
- Establish branching rules on solutions of this relaxation.

The convex hull relaxations of a square term x_1^2



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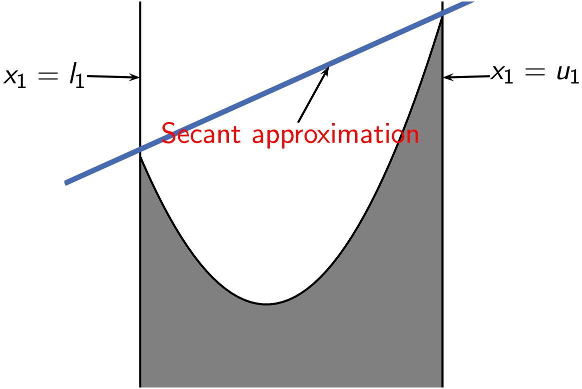

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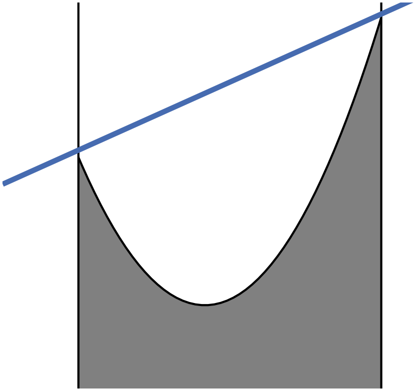
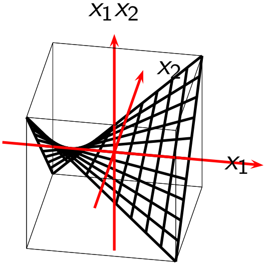


$$x_1^2 \leq y_{ii}^+ := (l_1 + u_1)x_1 - l_1 u_1$$

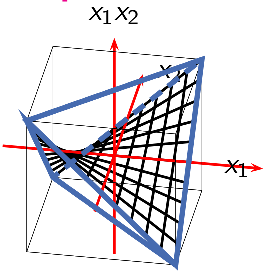
The convex hull relaxations of a single product $x_1 x_2$ [McCormick, 1976]



The convex hull relaxations of a single product $x_1 x_2$ [McCormick, 1976]

$$x_1 x_2 \geq y_{12}^{-} := \max \left\{ \begin{array}{r} u_2 x_1 + u_1 x_2 - u_1 u_2 \\ l_2 x_1 + l_1 x_2 - l_1 l_2 \end{array} \right\}$$

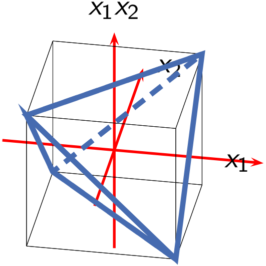
$$x_1 x_2 \leq y_{12}^{+} := \min \left\{ \begin{array}{l} u_2 x_1 + l_1 x_2 - l_1 u_2 \\ l_2 x_1 + u_1 x_2 - u_1 l_2 \end{array} \right\}$$



The convex hull relaxations of a single product x_1x_2 [McCormick, 1976]

$$x_1x_2 \geq y_{12}^- := \max \begin{Bmatrix} u_2x_1 + u_1x_2 - u_1u_2 \\ l_2x_1 + l_1x_2 - l_1l_2 \end{Bmatrix}$$

$$x_1x_2 \leq y_{12}^+ := \min \begin{Bmatrix} u_2x_1 + l_1x_2 - l_1u_2 \\ l_2x_1 + u_1x_2 - u_1l_2 \end{Bmatrix}$$



- Depending on the sign of q_{ij} we only need y^+ or y^-.
- For simplicity, we assume we put all in the remainder.

- Let $Q = P + \tilde{Q}$ with P the diagonal psd matrix containing $q_{ii} > 0$.

$$
\begin{aligned}
& \min \ \frac{1}{2}x^T P x + \frac{1}{2}x^T \tilde{Q} x + c^T x \\
& s.t. \\
& Ax = b \\
& x_j \in \mathbb{Z} \quad j = 1, \ldots, p \\
& l \leq x \leq u
\end{aligned}
\qquad \text{(MIQP)}
$$

- Let $Q = P + \tilde{Q}$ with P the diagonal psd matrix containing $q_{ii} > 0$.
- Add one $y_{ij} = x_i x_j$ variable for each non-zero entry q_{ij} of $\tilde{Q}$.

$$
\begin{aligned}
& \min \ \frac{1}{2} x^T P x + \frac{1}{2} \langle \tilde{Q}, Y \rangle + c^T x \\
& s.t. \\
& Ax = b \\
& x_j \in \mathbb{Z} \quad j = 1, \ldots, p \\
& Y = x x^T \\
& l \le x \le u
\end{aligned}
\qquad \text{(MIQP)}
$$

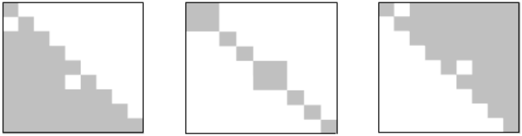
($\langle Q, Y \rangle = \sum_{i,j} q_{ij} y_{ij}$)

- Let $Q = P + \tilde{Q}$ with P the diagonal psd matrix containing $q_{ii} > 0$.
- Add one $y_{ij} = x_i x_j$ variable for each non-zero entry q_{ij} of $\tilde{Q}$.
- Relax $y_{ij} = x_i x_j$ using McCormik and Secant approximations.

$$
\begin{aligned}
& \min \ \frac{1}{2} x^T P x + \frac{1}{2} \langle \tilde{Q}, Y \rangle + c^T x \\
& s.t. \\
& Ax = b \\
& x_j \in \mathbb{Z} \quad j = 1, \ldots, p \\
& y_{ij}^- \le y_{ij} \le y_{ij}^+ \\
& y_{ii} \le y_{ii}^+ \\
& l \le x \le u
\end{aligned}
\tag{q-MIQP}
$$

Factorizations of Q

- CPLEX own block indefinite decomposition: M and B such that M 2-block triangular and B 2-blocks diagonal with $Q = M^T BM$



- Reformulate $x^T Qx$ using additional variables z so that $z^T Dz = x^T Bx$ and D diagonal. Let L, D give the spectral decomposition of B, $z = L\zeta$, $\zeta = Mx$.

(For simplicity assume $z = Lx$ gives the system we want)

Use a decomposition to get $z = Lx$ and $z^T Dz = x^T Qx$ and do the same steps as before (but more simple)....

$$\begin{aligned} &\min \ \frac{1}{2} z^T Dz + c^T x \\ &s.t. \\ &Ax = b, Lx = z \\ &x_j \in \mathbb{Z} \qquad\qquad j = 1, \ldots, p \\ &l \leq x \leq u \end{aligned} \tag{MIQP}$$

Use a decomposition to get $z = Lx$ and $z^T Dz = x^T Qx$ and do the same steps as before (but more simple)....

- Let $D = D^+ - D^-$ with $D^\pm$ diagonal psd matrices.

$$\begin{aligned}
& \min \ \frac{1}{2}(z^T D^+ z - z^T D^- z) + c^T x \\
& s.t. \\
& Ax = b, Lx = z \\
& x_j \in \mathbb{Z} \qquad\qquad j = 1, \ldots, p \\
& l \leq x \leq u
\end{aligned} \qquad \text{(MIQP)}$$

Factorized Eigenvector space reformulation and relaxation

Use a decomposition to get $z = Lx$ and $z^T Dz = x^T Qx$ and do the same steps as before (but more simple)....

- Let $D = D^+ - D^-$ with $D^\pm$ diagonal psd matrices.
- Add $y_{ii} \leq z^2$ variable for each non-zero of D^-.

$$
\begin{aligned}
\min\ & \frac{1}{2} z^T D^+ z - \sum_{i=1}^{n} \frac{d_{ii}}{2} y_{ii} + c^T x \\
s.t. & \\
& Ax = b, Lx = z \\
& x_j \in \mathbb{Z} \qquad j = 1, \ldots, p \\
& y_{ii} \leq z_i^2 \\
& l \leq x \leq u
\end{aligned}
\tag{MIQP}
$$

Use a decomposition to get $z = Lx$ and $z^T Dz = x^T Qx$ and do the same steps as before (but more simple)....

- Let $D = D^+ - D^-$ with $D^\pm$ diagonal psd matrices.
- Add $y_{ii} \leq z^2$ variable for each non-zero of D^-.
- Infer finite bounds, l^z, u^z for z and relax $y_{ii} \leq z_i^2$ using Secant approximations.

$$
\begin{aligned}
& \min \ \frac{1}{2} z^T D^+ z - \sum_{i=1}^{n} \frac{d_{ii}}{2} y_{ii} + c^T x \\
& s.t. \\
& Ax = b, Lx = z \\
& x_j \in \mathbb{Z} \qquad\qquad j = 1, \ldots, p \\
& y_{ii} \leq y_{ii}^+ \\
& l \leq x \leq u, l^z \leq z \leq u^z
\end{aligned}
\qquad \text{(ev-MIQP)}
$$

Notes on the two relaxations

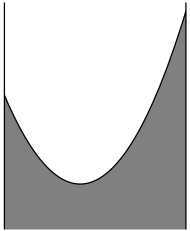
The steps are almost the same.

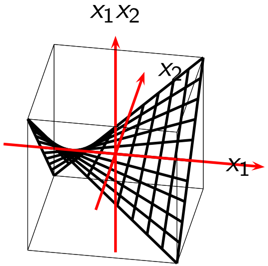
- If Q is diagonal the two relaxations are identical.
- In general they are not comparable.
- If $Q \succeq 0$, EV-space is better it preserves convexity.
- Q-space gives a surprisingly good approximation. Namely, [Luedtke et al., 2012] show that, if Q has a 0 diagonal, for the box QP: $\min\{x^T Qx : 0 \leq x \leq 1\}$:
  - if $Q \geq 0$ the approximation is within a factor 2:
  - if $Q \not\geq 0$ the approximation is within a factor of # nnz in Q (conjecture it is better)
- Many ways to do better splittings of Q, for example, with SDP [Billionnet et al., 2012].

CPLEX current strategy

- Uses EV-space if problem looks almost convex.

- Let $(\overline{x}, \overline{y})$ be the solution of the chosen QP relaxation after presolve/cutting. And assume $x_j \in \mathbb{Z}$, $j = 1, \ldots, p$.
- If $\exists \overline{y}_{ij} \neq \overline{x}_i \overline{x}_j$, $(\overline{x}, \overline{y})$ is not a solution of the problem and we need to branch.
- Pick an index i, choose a value θ between $\frac{l_i + u_i}{2}$ and $\overline{x}_i$.
- Branch by changing the bound to θ and updating all Secant and McCormick approximations involving this bound.





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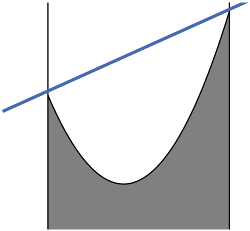

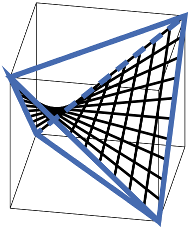

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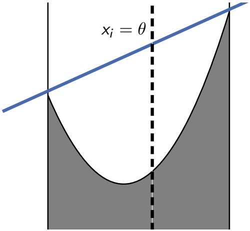

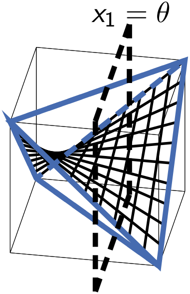

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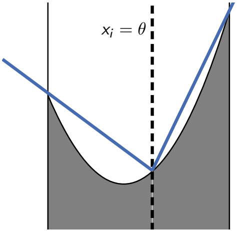
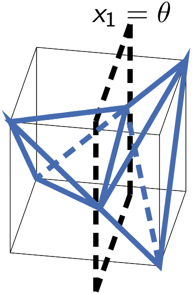




Other ingredients

- Convex QP relaxation solved by a QP simplex.
- Interior point solver for improving incumbents.
- Bound strengthening based on the KKT system.
- Linearize completely parts of the problem involving binary variables.
- Heuristic detection of unbounded problems.
- Multi-threaded.

- Try to bound all auxiliary variables with a basic presolve.
- If not possible, do it by solving LPs.
- If there is an unbounded direction r look at its cost $r^T Qr$:
  - If $r^T Qr < 0$: problem is unbounded,
  - If $r^T Qr \geq 0$: relaxation is unbounded but cannot conclude on problem status, return `RELAXATION_UNBOUNDED`.
- (Very easy to construct examples where can't conclude).

[Hu et al., 2012]

- Propose a KKT system that detects unbounded problems correctly.
- Use a combinatorial Benders approach to solve it.

Test set

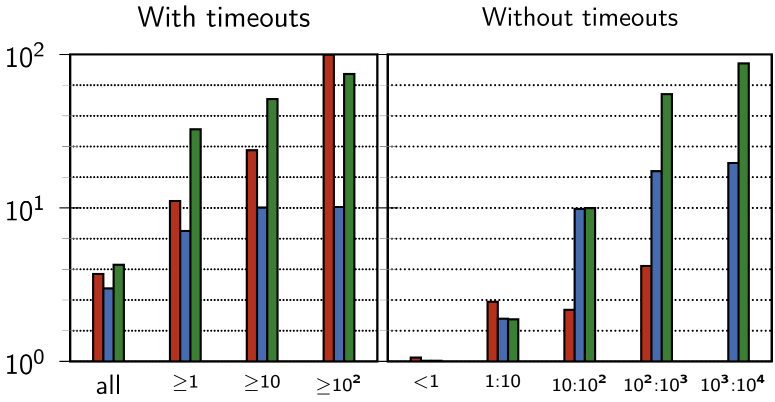
390 models

- Internal nonconvex MIQP (with three variants: original, 50% integer relaxed, 100 % relaxed).
- GAMS Globallib
- minlp.org, Box-QP, Tardella instances, ...
- CUTEr problems with flipped objective

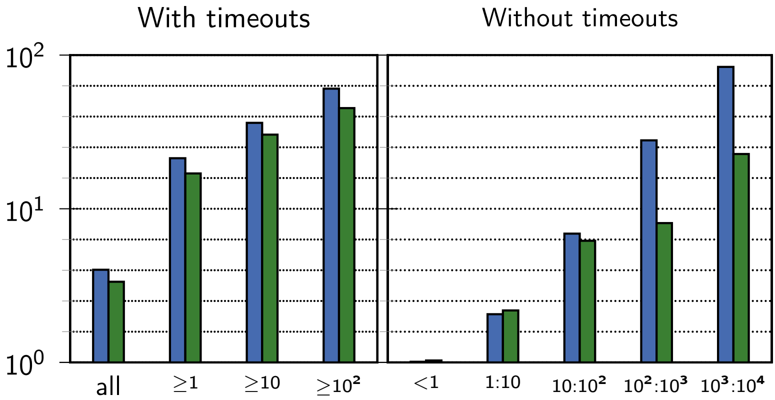
Experiments

- Not really any other solver aimed specifically at nonconvex MIQP.
- Compare with SCIP 3.0.1 [Vigerske, 2013] and Couenne 0.4 [Belotti et al., 2009] using 1 thread.
- Compare CPLEX with 1 and 4 threads.
- Time limit of 3 hours.

Comparison with SCIP on different test-sets



- Pure 0-1 models. Timeouts: SCIP 5.
- Mixed 0-1 models. Timeouts: CPLEX 2 , SCIP 2.
- Continuous and general integers. Timeouts: CPLEX 1, SCIP 29.

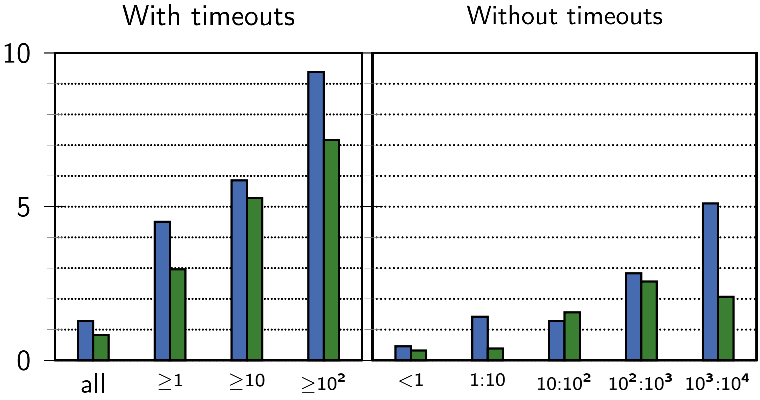


SCIP. 36 timeouts, 5 failures.

Couenne. 22 timeouts, 47 failures

CPLEX 3 timeouts and 7 failures.

CPLEX vs SCIP vs Couenne: Nodes



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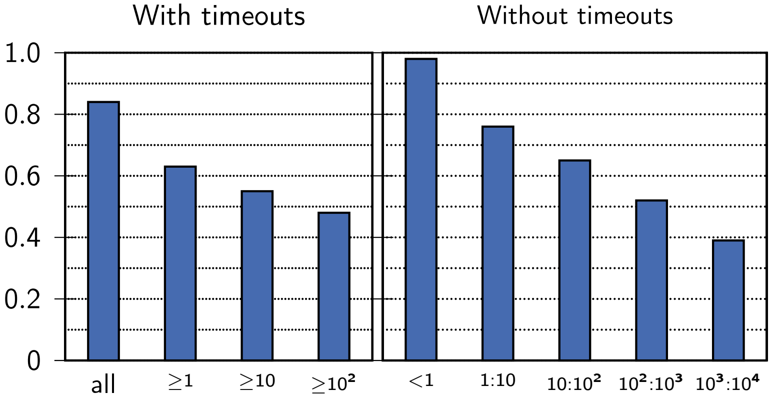
CPLEX only, 1 versus 4 threads on computing time



4 models not solved with 1 threads solved with 4.

Agenda

- The convex case
  - Main algorithmic approaches.
  - Glimpse at computation.
- A step into nonconvexity
  - MIQP with CPLEX 12.6.
  - Basic setup of a spatial branch-and-bound.
  - Computational results.
- Arbitrary selection of more advanced topics
  - Separability.
  - Disjunctive Cuts.
- Conclusions
- Problem Session
- Bibliography

- Preprocessing/Modeling:
  - separability [Hijazi et al., 14]
  - perspective formulations [Frangioni and Gentile, 2006, Günlük and Linderoth, 2008]
  - propagation [Vigerske, 2013]
- Node relaxations/Branching:
  - exploiting QP relaxation in strong-branching [Bonami et al., 2013] item divings [Mahajan et al., 2012]
- Primal Heuristics:
  - Feasibility Pumps [Bonami et al., 2009],
  - Undercover [Berthold and Gleixner, 2013]
- Cuts:
  - disjunctive cuts [Kılınc et al., 2011, Bonami, 2011],

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & g_i(x) \leq 0 & i = 1, \ldots, m \\ & x \in X & \\ & x_j \in \mathbb{Z} & j = 1, \ldots, p \\ & l \leq x \leq u & \end{array} \qquad \text{(sMINLP)}$$

- For $i = 1, \ldots, m$, $g_i : X \to R$ are *convex separable*:

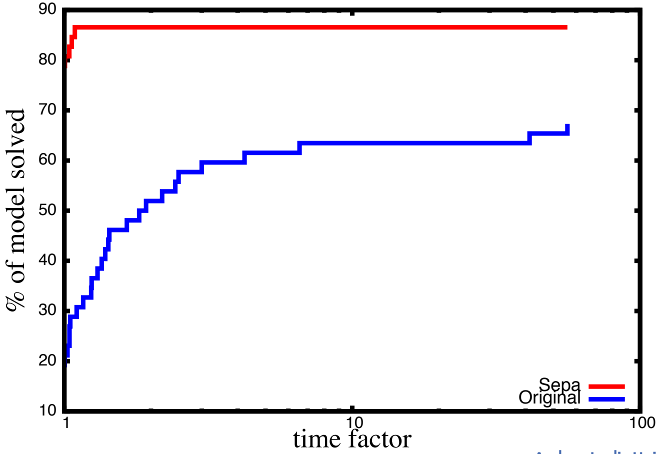
$$g_i(x) = \sum_{j=1}^{n} g_{ij}(x_j)$$

with $g_{ij} : [l_j, u_j] \to \mathbb{R}$ convex.

Introduce one variable y_{ij} for each elementary function:

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & \sum_{j=1}^{n} y_{ij} \le 0 && i = 1, \dots, m, \\
& g_{ij}(x_j) \le y_{ij} && \begin{array}{l} i = 1, \dots, m, \\ j = 1, \dots, n, \end{array} \\
& x \in X, \\
& x_i \in \mathbb{Z} && i = 1, \dots, p, \\
& l \le x \le u.
\end{aligned}
\qquad \text{(sMINLP}^*\text{)}
$$

- In the standard benchmark for MICP, out of 100+ instances, 8 are not directly separable.
- Constructing separated formulations on a subset of 47 instances gives a 3x speed up: [Hijazi et al., 14].



- Cuts are an essential component of MILP solvers.
- Of course one can always apply MILP cuts to a linear OA of MICP.
- How can we generate cuts that also exploit nonlinear constraints?
- Can we generate better cuts by looking directly at nonlinear functions?
- A partial answer: as long as the cut generated is linear it could also have been obtained from a linear outer approximation.
- In the past three years, tremendous activity towards conic cuts for conic programming but no general method yet, and no striking computational results.

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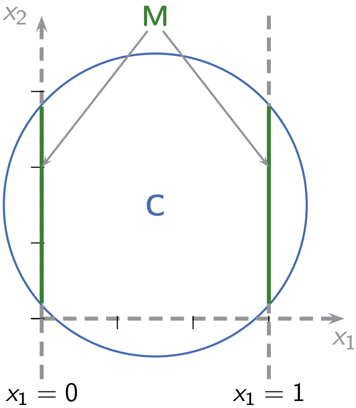
- In the past three years, tremendous activity towards conic cuts for conic programming but no general method yet, and no striking computational results.

Consider $\mathbf{C}$ and $\mathbf{M} := \mathbf{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.
Let $k \leq p$, $\pi \in \mathbb{Z}$ and

$$\mathbf{C}^{(\pi,\pi_0)} := \text{conv}\Big(\mathbf{C} \cap (\{x : x_k \leq \pi\} \cup \{x : x \geq \pi + 1\})\Big).$$

(clearly $\mathbf{M} \subseteq \mathbf{C}^{(\pi,\pi_0)} \subseteq \mathbf{C}$).



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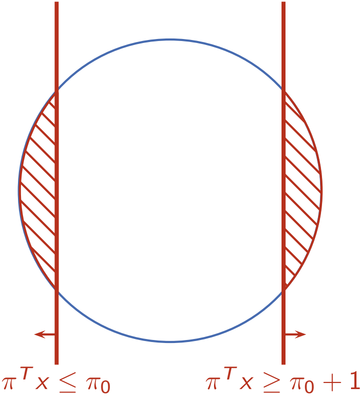

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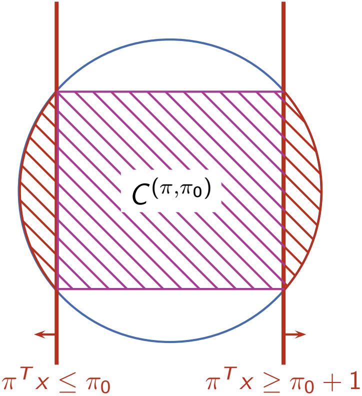

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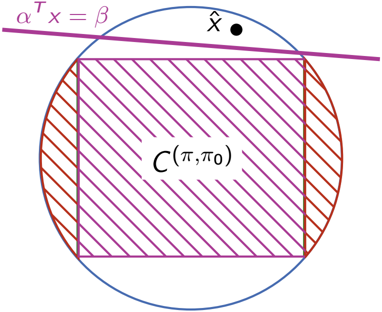
(clearly $\mathbf{M} \subseteq \mathbf{C}^{(\pi,\pi_0)} \subseteq \mathbf{C}$).

In the remainder, $\hat{x}$ is the point to separate, $\hat{x}_k \in]0, 1[$ $(k \leq p)$, and $\pi = 0$



Consider C a polyhedron $\{x : Ax = b, x \geq 0\}$

Cut Generation LP

$\hat{x} \in C$ is separated using disjunctive programming:

$$
\begin{aligned}
& \min \alpha^T \hat{x} - \beta \\
& s.t. : \\
& \alpha = u^T A + s - u_0 e_k, \qquad \alpha = v^T A + t + v_0 e_k, \\
& \beta = u^T b, \qquad \beta = v^T b + v_0, \\
& \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}, \; u, v \in \mathbb{R}^m, \; s, \; t \in \mathbb{R}^n_+, \; u_0, v_0 \in \mathbb{R}_+
\end{aligned}
\tag{CGLP}
$$

Goal: build a linear OA from which a "best" cut can be deduced using CGLP.

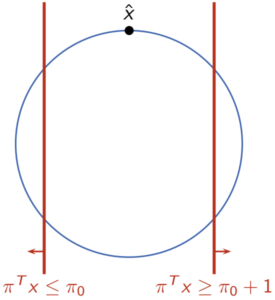
Using only LP [Kılınc et al., 2011].

1. Start with any linear OA of C
2. Solve CGLP. If no cut is found.
3. Deduce from dual of CGLP two points such that $\hat{x} = \lambda x^1 + (1 - \lambda)x^0$ and satisfying the disjunction.
4. If point(s) not in C generate new OA and goto 2, otherwise use the cut.

Using NLP [Bonami, 2011]

1. Solve a single NLP that tells if $\hat{x}$ is in the split relaxation.
2. If not, deduce from solution two points such that $\hat{x} = \lambda x^1 + (1 - \lambda)x^0$ and closest to satisfy the disjunction.
3. Build OA around these two points.
4. Solve CGLP and get the cut.

Goal: build a linear OA from which a "best" cut can be deduced using CGLP.



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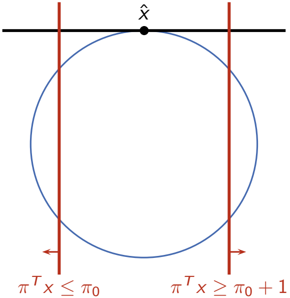

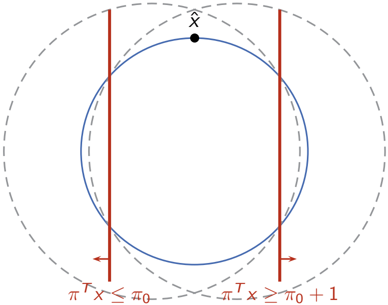

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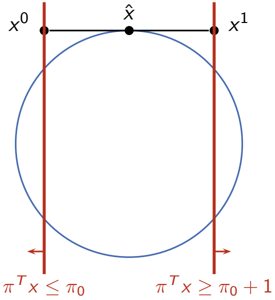

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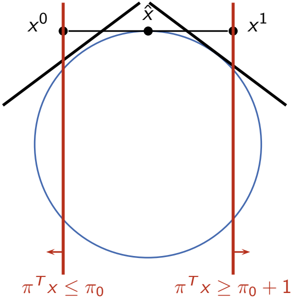

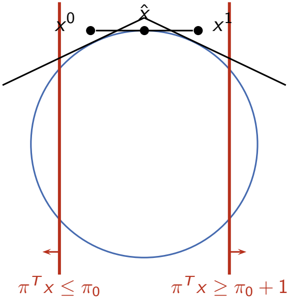

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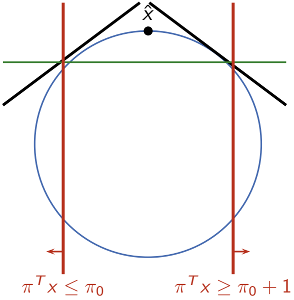

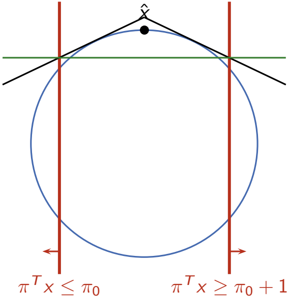

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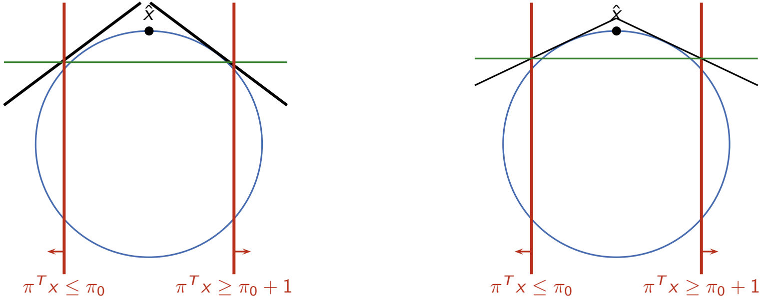


Andrea Lodi, University of Bologna

- [Kılınc et al., 2011] report a speedup of 3 on a set of "hard" instances with the NLP/LP FilMINT.
- [Bonami, 2011] report a speedup of 24 % on nontrivial instances with NLP B&B.
- In both cases, some instances not solved without these cuts are then solved within seconds.

Combination with separability[Kılınç, 2011]

Even better results are obtained by combining the extended formulation trick for separability and these cuts.

		Original		Extended	
	n	gap closed	sol time	gap closed	sol time
Batch	10	58.40	376.2	68.77	58.7
Markowitz	10	0.00	$>$ 10 800	98.07	1 262
SLay	14	68.50	36	86.08	5
uflquad	15	10.85	784	96.25	145

- MINLP is still very challenging and not well solved.
- In the last three years:
  - SCIP [Vigerske, 2013]
  - MINOTAUR [Leyffer et al., 2012]
  - GLOMIQO/ANTIGONE [Misener and Floudas, 2013]

  (each brought tremendous improvement over the state of the art).
- Commercial vendors are also moving.
- Good solvers need good test-sets:
  - www.minlp.org: repository of models.
  - more is needed.

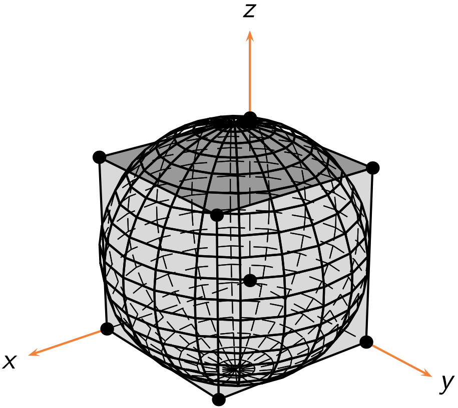
- The convex case
  - Main algorithmic approaches.
  - Glimpse at computation.
- A step into nonconvexity
  - MIQP with CPLEX 12.6.
  - Basic setup of a spatial branch-and-bound.
  - Computational results.
- Arbitrary selection of more advanced topics
  - Separability.
  - Disjunctive Cuts.
- Conclusions
- Problem Session
- Bibliography

Consider the following family of convex MINLPs:

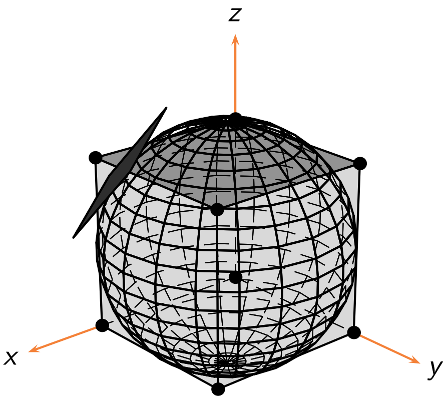
$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \sum_{i=1}^{n} \left(x_i - \frac{1}{2}\right)^2 \leq \frac{n-1}{4} \\ & x \in \mathbb{Z}^n \end{array} \tag{1}$$

(1) is infeasible:

- The ball is too small to contain integer points.
- It is large enough to touch every edge of the hypercube.



- No OA constraint can cut 2 vertices of the hypercube.
  - If an inequality cuts two vertices, it cuts the segment joining them. This cannot be: the ball has non-empty intersection with any such segment.



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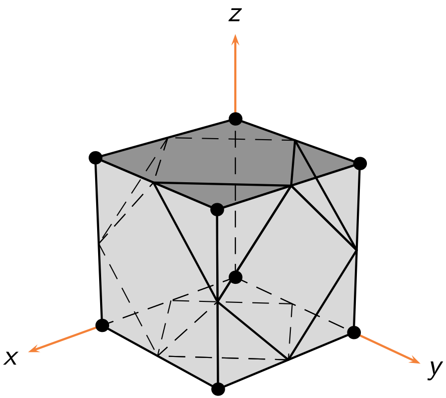

 Andrea Lodi, University of Bologna

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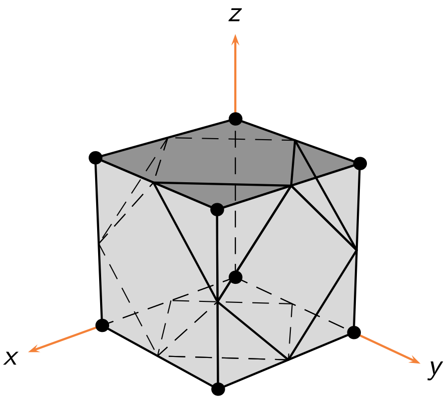

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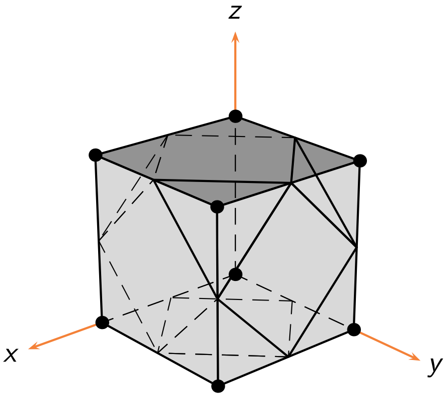

- Note: NLP branch-and-bound also enumerates at least 2^n integer sols.

		CPLEX	SCIP 2.1	B-OA	B-Hyb
n	2^n	nodes	nodes	OA it.	nodes
10	1,024	2,047	720	1,105	11,156
15	32,768	65,535	31,993	...	947,014
20	1,048,576	2,097,151	1,216,354	...	...

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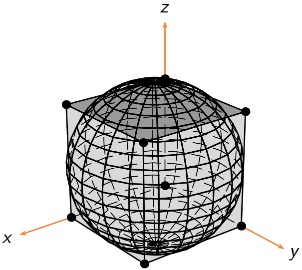
Remark

- Problem is simple for CPLEX/SCIP if variables are $0-1$: replace x_i^2 by x_i, the contradiction $\frac{n}{4} \leq \frac{n-1}{4}$ follows.
- SCIP > 3.0 applies tricks and solves it in a blink.

Extended formulation of (1)



$$\begin{aligned} \min\ & c^T x \\ \text{s.t.} & \sum_{i=1}^{n} y_i \le (n-1)/4 \\ & (x_i - 0.5)^2 \le y_i \qquad i = 1, \dots, n \\ & x \in \mathbb{Z}^n. \end{aligned} \tag{2}$$

Its outer approximation

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.} & \sum_{i=1}^{n} y_i \le (n-1)/4 \\ & 2\left(\overline{x}_i^k - 0.5\right)(x_i - \overline{x}_i^k) + \left(\overline{x}_i^k - 0.5\right)^2 \le y_i \qquad \begin{matrix} i = 1, \dots, n \\ k = 1, \dots, K \end{matrix} \\ & x \in \mathbb{Z}^n \end{aligned}$$

Application to (1) [Hijazi et al., 14]

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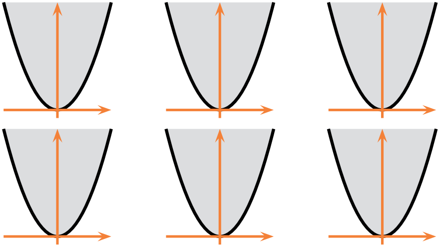

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Extended formulation of (1)

min $c^T x$

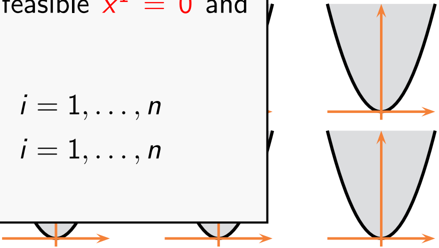
s.t. $\sum_{i=1}^{n}$

$(x_i$ -

$x \in$

2 points suffice to make it infeasible $\overline{x}^1 = 0$ and $\overline{x}^2 = 1$:

$$-x_i + 0.25 \leq y_i \qquad i = 1, \dots, n$$
$$x_i - 0.75 + \leq y_i \qquad i = 1, \dots, n$$



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